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Wittgenstein, Mathematics and the Temporality of Technique

One of the stated commitments of the later Wittgenstein' philosophy is that, just as philosophy must not in any way "interfere" with the practice of mathematicians, conversely and equally, "no mathematical discovery" can by itself "advance" philosophy in its quest to clarify the forms of our lives and language.[1] It would thus appear *ab initio* that, for Wittgenstein, the mathematician and the philosopher of mathematics, operating with different methods and in distinct regions of inquiry and insight, have very little to say to each other. On the other hand, however, Wittgenstein is committed equally strongly to the idea that philosophy can and *should* take a different kind of interest in mathematics: not as a body of results to be explicated or methods to be emulated, but as a set of techniques or practices *within* human life, to be understood in general terms only in the *context* of related practices that are not simply or exclusively mathematical, and thereby as illuminating *our* practices and ways of life, much more broadly. This includes the characteristic practices of mathematics "itself" – activities such as calculating and problem-solving, developing proofs and making conjectures. But it also, crucially, includes those practices that characterize our practical, lived, social, emotional and educational experience much more generally -- practices, for example, of teaching and learning, of understanding and being convinced, of "seeing" a relationship and "knowing" what is the right way to proceed. The philosopher's interest in these practices, and in particular in the ways that they are involved in (what we call) "doing" mathematics, extends to the illumination of what is meant by (or what we understand by) such "ordinary" phenomena and experience as those of: following a rule; practicing a regular method; developing a technique; arguing rationally for a conclusion; and convincing someone of something (whether by means of a "formal" or "informal" "proof.") With respect to each of these, Wittgenstein argues, the philosopher's attention to mathematical practice provides a decisive guideline for the broader kinds of clarification and illumination that philosophical reflection itself produces more generally. It does so, in part, by directing our attention to those features of (specifically)

[1] PI, 124.

mathematical practice that mark its role within the broader and multiple contexts of what we may call, using Wittgenstein's terminology, our collective and shared human and linguistic "form of life".[2]

Discussion of Wittgenstein's commitment to the inseparability of mathematics from human life has often taken the form of whether and to what extent it makes Wittgenstein an "anti-realist" about mathematical truths. Here, the focus is on what Wittgenstein thinks about the "status" of mathematical truths or entities, given that he thinks they *do* depend on "our" practices in some important way. Commentators have argued, for instance, about whether this commitment involves rejecting "Platonism" about mathematical truths or entities, or whether it means that he thinks that these depend on the contingencies of human societies or specific and historically variable cultures. In this paper, though, I take a different tack, arguing that Wittgenstein points to a concept of mathematical *technique* that is itself, and recognizably, both fully integrated into "human" life and *also* (nevertheless) fully and genuinely "mathematical." What is most important here is to see how a mathematical technique is *irreducible to* the "mechanical" application of a rule, *while still being fully* "mathematical" in the sense that it itself defines the kind of "access" and "availability" which mathematics and (what we may call) "mathematical entities" have for us. I argue that this conception of technique in the context of a human life is also intimately and irreducibly linked to the experience, reality, and (perhaps most importantly), the *temporality* of mathematical teaching, learning, insight and discovery, and cannot be separated from these contexts, even in principle.

I

In a much-discussed remark, written in 1944, from the *Remarks on the Foundations of Mathematics*, Wittgenstein considers, as he repeatedly does in the *RFM*, the question whether a string of 3 sequential 7s occurs in the decimal expansion of π (asked before we have actually found such a string by means of calculation):

[2] For the terminology of "form of life" [Lebensform], see, e.g., π 23: "…the *speaking* of language is part of an activity, or of a form of life."; and π 241: "'So you are saying that human agreement decides what is true and what is false?' – What is true or false is what human beings *say*; and it is in their *language* that human beings agree. This is agreement not in opinions, but in form of life." For a related development, drawing on some mathematical examples, see Livingston (2012), chapters 1 and 6.

> … What harm is done e.g. by saying that God knows *all* irrational numbers? Or: that they are already all there, even though we only know certain of them? Why are these pictures not harmless?
>
> For one thing, they hide certain problems. –
>
> Suppose that people go on and on calculating the expansion of **π.** So God, who knows everything, knows whether they will have reached '777' by the end of the world. But can his *omniscience* decide whether they *would* have reached it after the end of the world? It cannot. I want to say: Even God can determine something mathematical only by mathematics. Even for him the mere rule of expansion cannot decide anything that it does not decide for us.
>
> We might put it like this: if the rule for the expansion has been given us, a *calculation* can tell us that there is a '2' at the fifth place. Could God have known this, without the calculation, purely from the rule of expansion? I want to say: No.[3]

One commentator who has taken this remark to involve an "anti-realist" attitude toward mathematical truth is Hilary Putnam. On Putnam's reading of it in "Wittgenstein, Realism, and Mathematics," it shows that Wittgenstein held, at least at this time, that "a mathematical proposition cannot be true unless we can *decide* that it is true on the basis of a proof or calculation of some kind."[4] In particular, Putnam takes Wittgenstein to be here asserting that – in case the world ends without *our* having determined whether or not the sequence of 7s occurs – "the statement that 777 occurs in the expansion is neither true nor false…"[5] This view, as Putnam interprets it, is itself based in considerations about the way mathematical truth depends on what we *can* do – that is, what we are *in fact* able to establish, calculate or verify "by the end of the world." In limiting truth in this way, thus amounts to a "mathematical form of verificationism" on Putnam's reading, comparable in some respects with the verificationist positions of mathematical intuitionists and logical empiricists.[6]

[3] *RFM*, VII-41 (p. 408).
[4] Putnam (2002), p. 421.
[5] Putnam (2002), pp. 431-32.
[6] Putnam (2002), p. 421

As Putnam notes, Wittgenstein develops a parallel example (the existence of a string of four 7s in the expansion of pi), but now without reference to either "God's omniscience" or the *actual* extent of human calculation by "the end of the world," in the *Philosophical Investigations*.[7] Instead, what Wittgenstein says in the parallel remark in the *Investigations* is that the *question* of the occurrence of the string of 7s in the expansion of π is, at any rate, "an English sentence" and one we understand. We understand and can explain what it would *mean* (for example) for '415' to occur in the expansion, "and similar things;" thus, Wittgenstein says here, "our understanding of that question reaches just so far, one may say, as such explanations reach."[8] On this sort of view, even if one does not introduce speculations about the *grounding* of the decimal expansion "itself" in our practices of calculating it, still the meaningfulness of our *understanding* of the question about whether the string of 7s occurs remains in an important way dependent on our *understanding* of what (exactly) an infinite decimal sequence is, as developed from a rule. This understanding has its place within, and is "constrained by", not only the calculation itself, but also practices of explanation, communication, reflection and comprehension that are not simply or exclusively mathematical, but are situated much more broadly within our ordinary linguistic understanding of 'what it is' to follow a rule that "determines" an infinite sequence, itself.[9]

Another commentator who has read the implication of these and other remarks as "anti-realist" is Michael Dummett. In his initial (1959) review of Wittgenstein's *Remarks on the Foundations of Mathematics*, Dummett somewhat famously read Wittgenstein as committed overall to a particularly strong form of conventionalist verificationism, what he called "full-blooded" conventionalism. On this sort of view as Dummett explicates it, "all necessity is imposed by us not on reality, but upon our language;" and indeed a mathematical statement is "necessary by virtue of our having chosen not to count anything as falsifying it."[10] This implies broadly that the "recognition" of mathematical necessity is in fact only a recognition of the immediate or mediate implications of linguistic conventions that "we

[7] PI, 516.
[8] PI, 516.
[9] On Putnam's reading (Putnam, 2002, pp. 438-40), the later remark in the *Philosophical Investigations* represents a retreat from the anti-realism that is (on his reading) exhibited in the 1944 RFM remark. For Putnam, this is shown by the fact that the latter remark says nothing about God or omniscience, but rather only about "our" understanding; thus what Wittgenstein is rejecting in the latter remark, according to Putnam, is not realism about mathematics (in some sense of "realism") but rather the view that there can be a "non-mathematical" explanation for the truth of mathematical truths. Although this is, on the current reading, in fact an important strand of the argument of both passages, what is less evident is that even the first (RFM) remark *must* be taken as maintaining an "anti-realist" position in any important sense (if, as both remarks suggest, the centrality of mathematical *practice* for the intelligibility of mathematical questions is maintained).
[10] Dummett (1959), p. 169.

have adopted;" thus the conventionalist (in this sense) holds that there is "nothing to" either the necessity or the truth of mathematical statements in general that goes beyond the implications of these self-chosen "conventions". What is more, Dummett argues, Wittgenstein's peculiarly "full-blooded" variety of this conventionalism consists in denying *even* that the implications of convention may be "mediate" in not following directly from these choices. For Wittgenstein as Dummett reads him "the logical necessity of any statement is always the *direct* expression of a linguistic convention," indeed, our "decision" to treat *just that* statement as unassailable.[11] This is because (for Wittgenstein as Dummett reads him), if we were to characterize our judgments of mathematical necessity as following from our initial conventions only mediately, we would still face the question of *how* those conventions are to be applied in the particular case, and would thereby have to appeal to an essential element of decision in each case anyway. This points, on the imputed interpretation, to the necessary and constitutive role of conventions formed *de novo* in each case of a new mathematical judgment; though these conventions may *subsequently* serve as standards for the repetition of just that judgment, they remain irreducibly our own, and have no deeper or extrinsic independent "grounding" in mathematical reality or the mathematical facts themselves. On the level of practice itself, it follows from this that (on the view imputed) there is nothing beyond our simply *doing* what we (in fact) do, in our ordinary practices (e.g. of counting, adding, etc.), that justifies or establishes its correctness.

As Dummett notes, the view thus attributed has highly implausible consequences on the level of mathematical practice itself. For example, as he argues, given "full-blooded" conventionalism, it is impossible for us to criticize the practices of a group of people who, after counting separately five boys and seven girls in a classroom, then subsequently re-count the group as a whole and come up with thirteen.[12] The most we can say is that given *our* practices and conventions, *we* would not count this way; but there is no intelligible neutral perspective from which we can say that they are in fact *wrong* to do so, or that they thereby get the facts or mathematical realities wrong. Dummett objects, for obvious reasons, to this kind of position. It is a clear and evident aspect of mathematical practice, whether "sophisticated" or not, that we *do* feel a kind of "responsibility" to the mathematical facts, and that we will subject to criticism those who we think have got them wrong. This is as much an aspect of everyday practices of counting and calculating with numbers as it is of "sophisticated" mathematical proof itself; if we do not at any rate experience the constraint of *feeling* responsible, whether in calculation or proof,

[11] Dummett (1959), p. 170.
[12] Dummett (1959), pp. 173-75.

to something that is not simply a matter of decision or arbitrary convention, we are not "doing" math – or engaging in the kinds of practices that we take doing math to be – as we *do* in fact engage in these practices in the course of our ordinary lives.

Although Putnam and Dummett disagree about the extent and endurance of Wittgenstein's conventionalism overall, they agree in thinking that Wittgenstein situates the *question* of the existence or determinacy of mathematical truths within a constitutive consideration of the extent and limits of *our* specific linguistic practices. In other words, the form of the question, seen this way, is: *given* that our practices (can) only go so far, what should we say about mathematical truths that outstrip them? Further, what links both of the interpretations, despite their differences, is the thought that mathematical truths are themselves determinately *constrained* by the limited *actual* extent of *our* specific "conventional" linguistic practices of decision, explanation, or calculation. So far as these practices go, we may talk without difficulty about mathematical truths or entities as "determinate," as "fixed," or as "real," but if we try to go any farther in our conception of them, we are subscribing to a misleading and dangerous "Platonizing" picture of mathematical reality, one which has no warrant in this reality (as, anyway, it is in any meaningful sense available) itself.

This thought has its broader setting in the context of familiar questions much pursued in the "philosophy of mathematics," questions such as: are mathematical truths "in us" or "out there"; are they "fixed" independently of our procedures for accessing them or our ways of knowing about them; are they, fundamentally, "created" or "discovered"? Now, my own view is that Wittgenstein's considered answer to all of *these* questions is "neither-nor;" and that his basic reasons for this answer turn on his nuanced conception of just *how* mathematics is 'integrated' into a human life. Rather than argue for this here, though, I will just try to show how this more nuanced conception fits within an idea of mathematical practice as shaped by, and in turn decisively shaping, a kind of life that we can recognize more broadly as "human" – and thus (a fortiori) *not* something that could be pursued, in any meaningful way, by something that does not live this kind of life, for example a machine or a computer. Putnam and Dummett's interpretations have it in common that they are intended primarily to address the question: what is mathematical *truth* and how is it "fixed" (if at all)? But Wittgenstein is not asking this question, or at any rate is not asking *only* this question. Just as much, he is asking about what *doing* mathematics is, and how it is related to what mathematics is "about". And he answers these latter questions, not with any specific overall account of the metaphysics of mathematical facts or truths, or the logical implications of their accessibility, but rather by adverting to the general and familiar circumstances in which practices such as counting, calculation, problem-solving and proof have their role within (what we already know as) our lives.

In this consideration, one concept and experience that has a particularly central significance (extending also far beyond just its implications for mathematical practice itself) is that of a (mathematical) *technique.* By means of such a technique – for example, the technique of calculation invoked in the 1944 remark about the expansion of π – one can arrive at a result which one recognizes as correct, and may be in a position to *use* this result (the approximation of π, say, to five places) for a particular purpose (to estimate, for instance, the area of a circular enclosure of a certain radius). This technique has its point and its purpose in the context of what *we* do with it and can be learned, taught, communicated and elaborated on in this way: as part of the specific "language game" of (for instance) calculative estimation of areas, but also as part of the *practices* that are called learning, teaching, and communicating mathematics. It is part of this learning and teaching that one learns that the technique "in principle" "goes on forever": that there is no end to "the decimal expansion of π" since the rule can always be applied to produce more digits, and in learning *this* a student learns also to make sense of the *question* about whether a string of 7s "appears" or not anywhere "in" the total expansion. But all of this is not to say that the expansion is as a whole "in" the (finitely stateable) rule itself, just "waiting" (as it were) to be unpacked from it. Rather, it is for Wittgenstein the irreducibility of the calculative technique *as* technique – its role and significance *within* a human life – that "gives" us whatever it does give us, whatever we can then use or articulate on the level of use for the diverse purposes to which we might put it.

This is not to say, of course, that we simply decide (or choose freely conventions that decide) what the value of the expansion will be, either at any particular place or in general. In pointing to the irreducibility of mathematical technique, Wittgenstein is raising the question what is it *for us* to follow a procedure or a rule, of how we are thereby "guided" and in what respect we are "free," of how our methods and procedures are rooted in the instances and practices of our lives, but *also* how they (nevertheless) give us the kind of insight and orientation that consists in knowing *how things are*. The point is that, on the one hand, it makes sense to say that our practice of calculation is *our* practice – that it has its point, its purpose, and its whole existence within the broader context of how we do it and what we do with it – but that on the other, it also makes sense to see its results as *completely* determinate, objective, and "fixed", even in their whole (infinite) extent. The difficulty of maintaining this position isn't that of accounting for how there can be so much as an objective "standard" for the correctness of our practices given that these practices are our own, but rather of showing *what* such a standard means from within these practices themselves. And this showing, as I have argued, makes essential use of the idea of a technique or a practice which is (on one hand) integrated into our lives, but on the other is "in principle" extensible, and responsible to what it is trying to illuminate.

How, then, should we read in this light the 1944 remark about God and the limitations of His omniscience? *One* of Wittgenstein's goals here is, obviously and admittedly, to challenge a "Platonistic" picture of mathematical truths according to which they are completely determinate and fixed in advance of anything like a calculation (or perhaps a proof), and thereby open in principle, as it were, to simple *inspection* by a superior entity. In denying that God, even if he knows *everything*, knows *without calculation* whether or not there is a string of 3 7's anywhere in the decimal expansion, Wittgenstein is evidently denying that there is any sense to be made, in other words, of the picture of the *whole* of the decimal expansion as (as it were) simply given to the deity, along with the knowledge *that it is* the whole correct expansion, in such a way that He could simply "look and see" whether the string occurs anywhere, or know this immediately *just* by being given the rule itself and without Himself calculating. But there is more to Wittgenstein's remark than *only* the rejection of this Platonist picture. There is also the reminder – equally obvious, perhaps, but difficult to hold in balance with the first, anti-Platonist point – that *there is* a procedure which suffices to calculate successively the digits of the expansion of π (and thus *eventually* to find a string of 3 7's, *if* any such string exists). This is just the calculative procedure that one normally uses, or that we can now program computers to follow out *much further* than we (given the *biological* limitations of our own finite capacities and limited lifetimes) can do. But the fact that such a procedure can be "automated" in such a way does not imply that it is not one whose scope, and significance, is to be understood from *within* its role in something like a human life.

In the very next remark of RFM, Wittgenstein emphasizes this essential embedding of the calculative procedure as an activity, within life and the more general "language-game" of following a rule itself:

> The concept of the rule for the formation of an infinite decimal is – of course – not a specifically mathematical one. It is a concept connected with a rigidly determined *activity* in human life. The concept of this rule is not more mathematical than that of: following the rule. Or again: this

> latter is not less sharply defined than the concept of such a rule itself. – For the expression of the rule and its sense is only a part of the language-game: following the rule.
>
> One has the *same* right to speak of such rules in general, as of the activities of following them.[13]

The upshot of these remarks, in their context, is not indeed to deny that the activity of calculating, for instance that of calculating the expansion of pi, is (as Wittgenstein says) a *rigidly* determined one. The procedure, in this case as in others, is not arbitrary; nor is it *just* a matter of convention, or decision, or just made up by us. Moreover, we have, even from within the procedure and the form of activity determined by it, an idea of what it means to talk about what "is" or "is not" within the decimal expansion as a whole. We can perfectly well understand, for instance, what it would mean for the sequence in question to be found, as we can also understand that the process *might* run on forever without finding it. It is, in this sense, and for this reason, perfectly legitimate to consider the principle of bivalence as applying to the expansion as a whole: either the string of 7s is "there" or it is "not there." Indeed, it is not clear what could be meant by denying this, once we have seen the rule as one that "goes on forever" in the way that it does.

What is disputed in the remarks, then, is not realism in general or realism about the expansion, but rather the misguided impression that realism itself demands that anyone – God included – could have access to the *whole* of what is produced by the procedure *without going through that procedure himself*. It is the procedure (or technique) that is irreducible, and it is also irreducible that it is something that is *gone through*, that is, something that is (can be) *practiced* by someone. It is immaterial to this whether the relevant "someone" is itself finite or infinite (whether, that is, its powers are conceived of as limited or unlimited); what matters is rather that these "powers" actually be applied: this is what Wittgenstein is calling proceeding "by means of mathematics" in the remark. This procedure, and others like it, are not less "rigidly determined" than we expect mathematical rules to be – that is, to say that they have a role in human life, and essentially so, is not to say that they do not determine *just* the results that they do, or that these will not be "objective" or fully "fixed." But equally it is not "determined" at all, except *as* an activity, a procedure that *we* can understand and "go through," something that is *done* in the context of a human life. The concept of this activity is not, as Wittgenstein says, *simply* a mathematical one, any more than the broader concept of "following a rule" which it exemplifies. But on the other

[13] RFM, VII-43 (p. 409).

hand, it is not therefore any the *less* “mathematical”; for it is in activities of this sort, and in the surrounding ones of learning, teaching, communicating and reasoning about them, that (in any meaningful sense) “doing” mathematics consists.

II

In the *Philosophical Investigations*, Wittgenstein devotes an extended consideration to the question of what it is to follow a rule. In this consideration, he often uses “mathematical” examples: not ones drawn from the practice of sophisticated mathematicians, but examples of the type that may arise in ordinary mathematical pedagogy at the grade school level. For example, a student is challenged to continue the series 1, 5, 11, 19, 29 by finding (as we might put it) its “rule”; at this point, Wittgenstein imagines, the student exclaims “Now I can go on!”[14] As Wittgenstein points out, the student’s understanding, which seems to occur *at* this moment, is consistent with a wide variety of possibilities about what may have occurred with him “mentally:” perhaps he has hit upon the formula: $a_n = n^2 + n - 1$; or perhaps he has simply observed the series of differences (4, 6, 8, 10), or perhaps neither of these happens: he just watches, says “Yes, I know *that* series” and continues it. Again, it is *also* coherent to suppose that any of these “mental” phenomena occur *without* the student in fact being able to continue it correctly (at this stage or any other): perhaps, for instance, the formula *does* occur to him but he still does not understand *how* to apply it.[15] What, then, are we to say “is” the understanding that occurs in the moment, the understanding by virtue of which he is able (if he is) to continue the series correctly and indefinitely?

In the more extended development of the “rule-following” considerations in the *Investigations*, Wittgenstein considers also examples that are recognizably “odd” from the perspective of “ordinary” ways of learning and responding to teaching. For instance, a student is taught to write the sequence with the rule +2 (2, 4, 6, 8, ..) , and does so correctly up to 1000; at this point, however, he writes 1000, 1004, 1008, 1012…[16] Wittgenstein’s point is not that we do, or should, ordinarily expect this kind of behavior, but rather that there are limits to what we *can* point to in “correcting” the student’s misapplication, and that there is no *single* or *simple* way to guarantee that a student will understand whatever we do point to in the way that *we* wish (and normally expect) him to. On the contrary, what

[14] PI, 151.
[15] PI, 152.
[16] PI, 185.

establishes that the student *has* understood correctly, and what constitutes understanding correctly – in ordinary circumstances – is not to be referred to his having any particular momentary experience, but rather to the complex of *circumstances* that surround any such experience in the broader context of the practices in which it takes place: here, indeed, the circumstances of teaching and learning, of coming to grasp and understand in the ways that we (in fact) do. These circumstances are more than just external accompaniments to the "language game" of learning and understanding how a rule "determines" a series: rater, they constitute this learning and understanding itself. And as such, they also, and equally, constitute "what it is" to *follow* a rule itself: what it is to "determine" or *see as* determinate what follows from it at any particular stage. Just as with the calculative case above, what is essential here is the idea of a technique which is fully, and irreducibly, *part* of a human life; but nevertheless is as completely "determinate" and whose following out is as "objective" as anything could be.

How, though, can the *whole* series, which after all is infinite in extent, ever actually *be* determined (objectively and fixedly) simply by something that happens (as it indeed does) at a particular moment, the moment of understanding or insight? The question is pressing, especially in view of the consideration that *any* finite item (for instance any finite segment of a series, or any symbolic expression of a rule "for" determining it) can be *variously* interpreted, and so cannot be seen as fixing all by itself its total infinite extent. At *PI* 213, an interlocutory voice responds to this kind of consideration by imagining that, where finite items are insufficient, an infinitely renewed or repeated *intuition* might do the trick of fixing the right interpretation across the extent of the series:

> 213. "But this initial segment of a series could obviously be variously interpreted (for example, by means of algebraic expressions), so you must first have chosen *one* such interpretation." -- Not at all! A doubt was possible in certain circumstances. But this is not to say that I did doubt, or even could doubt. (What is to be said about the psychological 'atmosphere' of a process is connected with that.)
>
> Only intuition could have removed this doubt? -- If intuition is an inner voice – how do I know *how* I am to follow it? And how do I know that it doesn't mislead me? For if it can guide me right, it can also guide me wrong.
>
> ((Intuition an unnecessary evasion.))

> 214. If an intuition is necessary for continuing the series 1 2 3 4 …, then also for continuing the series 2 2 2 2 …[17]

Wittgenstein's point is that the thought that the continuation of a series requires a new intuition at each instance would, if tenable, also apply to the seemingly most simple kind of rule, the one that requires only the infinite repetition of the same. And if intuition *here* functions as a kind of inner voice, then it also appears *even here* that it might mislead. But if even the infinite repetition of the same is not capable of "securing" the determinacy of the rule's extension, then nothing – at least nothing on the level of the internal or mental *accompaniments* of the actual practice and circumstances of learning and understanding – can be seen as doing so. What is rather to be seen as the necessary, as well as sufficient, condition for such understanding is the complex , various, and irreducible circumstances *in which* such a rule is normally understood, learned, and followed: the circumstances, that is, of the practices of "mathematical" teaching and learning within the context of a human life.

> 47. If a rule does not compel you, then you aren't *following* a rule.
>
> But how am I supposed to be following it; if I can after all follow it as I like?
>
> How am I supposed to follow a sign-post, if whatever I do is a way of following it?
>
> But, that everything can (also) be *interpreted* as following, doesn't mean that everything is following.
>
> But how then does the teacher interpret the rule for the pupil? (For he is certainly supposed to give it a particular interpretation.) Well, how but by means of words and training?
>
> And if the pupil reacts to it thus and thus; he possesses the rule inwardly.
>
> But *this* is important, namely that this reaction, which is our guarantee of understanding, presupposes as a surrounding particular circumstances, particular forms of life and speech. (As there is no such thing as a facial expression without a face.)

[17] PI, 213-14.

(This is an important movement of thought.)[18] (RFM pp. 413-14).

If Wittgenstein is right about this, what consequences follow for the learning and teaching of mathematics? At one level, in developing these examples, Wittgenstein is just calling attention to what actually does happen in the classroom, and to such ordinary and practically unavoidable circumstances of pedagogy as that a student may, on being given any kind of instruction, still fail to grasp its point; or that no single explanation or example of a method can be relied upon to produce understanding in *any* student or at *any* moment. But on a broader and more "philosophical" level, Wittgenstein's considerations show the infelicity of any conception of mathematical pedagogy that sees it as a matter *simply* of giving explanations or displaying regular methods at all. What his remarks seem to point to is that, since the understanding invoked and drawn upon in "knowing how to go on" is not just "there" in any given item or symbol, this understanding cannot be successfully produced, in general, by means of many of the practices that are in fact pervasively common in mathematics classrooms, especially at grade school levels: practices such as repeated drilling in mechanical methods, or rote memorization of formulas. If the infinite series of values "determinable" by means of a mathematical technique is not just available to be "read off" from Platonic heaven or reality in itself, then it is also not simply "available" in the symbolic expression of a rule, or the repetition of a mechanical procedure, either. Rather, if the teaching and learning of technique is irreducible in the way Wittgenstein suggests, what matters most is to preserve this irreducibility, to maintain it as an intrinsic and unavoidable part of the relationship between teacher and student involved in the practice of learning mathematics, and to understand the pedagogical experience as irreducibly part of the bearing of mathematics on the diverse circumstances of the individual and collective lives we lead, which is to say as irreducibly part of *mathematics* itself.

III

Drawing on Wittgenstein's remarks, I have argued for a conception of mathematical technique according to which it is both (on the one hand) situated in the complex context of human forms of life and (on the other) nevertheless *fully* determinate and "objective." I have further suggested that such a conception of technique, as fully and irreducibly "situated" within the finitude and temporality of a human life, can be seen as an irreducible and essential part of what mathematics *itself* is, so that it is not

[18] RFM, VII-47, pp. 413-14.

really possible to characterize “mathematical entities” or “truths” as they are even “in themselves” without reference to the complex and dynamic circumstances of a human life. This conception can itself seem mysterious, however, given typical and longstanding assumptions about the temporality of mathematics, assumptions that characterize mathematical facts or truths as “outside” time and thus immune to the possibility of change or transformation. As thus “outside” time and change but nevertheless available in a unitary and objective way to us, these facts or truths are also, on this traditional conception, in some sense “outside” the empirical world of facts and experience: they stand beyond or before the totality of such facts and are wholly isolated from the contingencies they involve. These assumptions already make it mysterious *how* an entity situated within time (such as ourselves) *can* (so much as) have any kind of “access” to them; and even granting some such “access,” they pose the problem of the temporal form of mathematical understanding itself. How (as we may put it) do we have, in finite temporality and at a distinct moment, a sufficient *insight* into the “whole” extent of an infinite mathematical structure? As I shall argue in this last section, this question (as Wittgenstein has pursued it) can be seen as marking in an interesting way a set of basic questions about the form of time as it is given or accessible to a human understanding, and as it can be seen as structuring the world as such.

As we have seen, in his remarks in *RFM* and the *PI*, Wittgenstein is concerned to refute a conception of the infinite totality of a mathematical series on which all of its members would be understandable as “just there”, all at once, as open (for instance) to simple *inspection* by a divinity or divine entity. To this he opposes the idea of a mathematical or calculative technique, a *doing* of something “by means of mathematics,” which essentially constitutes what it is to (be able to) know the series, and hence plays an *essential* role in (as we may put it) constituting the series itself. What is more, what is given in and through the technique is not to be seen as consisting simply in *any* totality of empirical facts about what in fact is calculated or discovered: Wittgenstein says that even if God were omniscient in the sense of knowing all of these facts, up to the “end of the world,” he could not know the answer to a question about what is in the decimal expansion without actually doing the calculation himself.

These remarks, of course, are not directly about time or the temporal form of the world. But their topic is nevertheless evidently very closely related to the question of the structure of time, and indeed to the question of the possibility of a realist position on it. This relation is marked not only in the intimate historical connections of philosophical thought about the subject matter of mathematics, since at least

Plato, with the idea of the timeless or extra-temporal, but more directly and immediately in the context of discussion that clearly shaped many of Wittgenstein's own ideas about mathematics and truth: that of the debate between formalists and intuitionists of the 1920s and early 1930s. In broader philosophical terms, both of these positions are notable for the particular kind of temporal perspective they involve. On the formalist's conception, the specification of a formal system along with its constitutive rules is itself enough to secure access (given at least its consistency) to the entire and infinite realm of what it discusses; this specification is in itself atemporal, and the rules are themselves to be finitely stateable in such a way as to be repeatable *ad infinitum*. Here, whereas the actual "application" of the rules (e.g. in a calculation) requires some sort of actual process or agency, the rules *in themselves* and what they determine are nevertheless essentially extra-temporal: independent and fixed with the specification of the formal system as the "ideal" system that it is. For the intuitionist, by contrast, there is always and irreducibility a *temporal* element in our mathematical cognition: we cannot take ourselves to genuinely have or understand a mathematical structure unless we do so by means of a finite, and temporal, intuition; and the being and existence of mathematical entities and phenomena is itself essentially dependent on the anterior and posterior structure of flowing time.[19]

Do Wittgenstein's remarks quoted above about mathematical practice and technique, then, show him to be a formalist or an intuitionist about mathematics and time? Again, the answer is "neither-nor," but we can see more clearly the temporal implications of what is in fact his position by considering how it might be generalized to provide a broader conception of the way that time itself is "given" to be thought and experienced. As we have seen above, Wittgenstein's purpose in the remark about God and omniscience is not necessarily to dispute that there is such a thing as "the decimal expansion of *pi*", and it is not to maintain the "anti-realist" position that there is no "fact of the matter" about whether the string of 7s appears in the expansion or not. What is disputed in the remarks is rather the misguided impression that realism itself demands that anyone – God included – could have access to the *whole* of what is produced by the procedure *without* going through that procedure itself. To dispute this is not to affirm a limitative finitism about the expansion of π or about mathematics in general, or to require anti-realism about such places in the decimal expansion as have not *yet* been determined by any human mathematician. But nor can it be that Wittgenstein here envisions an ultimate basis for *our* procedure of calculating the expansion of π in a kind of primitive temporal *intuition*, of the kind that Brouwer's

[19] For the terms of the dispute, see, e.g. Hilbert (1925) and Brouwer (1912).

intuitionism proposed at the ultimate (and ultimately temporal) basis of mathematical knowledge.[20] We know, biographically, that Wittgenstein was decisively influenced by Brouwer's intuitionism just before his return to philosophical activity in 1929; this influence leaves its mark, in particular, on some of the anti-realist and verificationist-seeming suggestions he makes about the relationship of proof and truth in mathematics during the 1930s. But there is also abundant evidence that he had abandoned both verificationism and intuitionist strictures (for instance against the use of the law of excluded middle) by the time he wrote most of the *Philosophical Investigations*.

The decisive consideration here, in fact, is the one already rehearsed above with reference to the question of continuing a regular series: if it is impossible to explain the basis for "going on" correctly with the expansion of a series *without* appealing to the directive provided by a primitive intuition, it is also impossible to explain it *by means of* such an appeal. This is because the appeal to a basic guiding intuition, as Wittgenstein points out, is simply (once more) the appeal to an essentially finite item, which does not settle its own interpretation, and so simply repeats the original problem. Analogously or homologously, if we cannot explain the givenness of time to our experience and thought *without* appealing to a basic intuitive giving, we cannot explain it *by* doing so either.

What remains nevertheless of Brouwer's idea of a constitutive dependence of mathematics on time, and does certainly influence Wittgenstein's remarks here, is the idea of the form of the "facts" as conditioned by the form of a possible understanding of them, and this form itself as being irreducibly temporal. The point is that we cannot conceive of these facts as "given" from wholly outside time, or presented as such to a being that exists outside it, except through a form which is itself in the relevant sense temporal, that of a calculative procedure. Rather, the necessity of conceiving the temporal facts from a position *within* time marks the form of these facts themselves, in such a way that it is not even so much as coherent to suppose that they could be conceived or perceived from beyond any such position.

If we were, then, to draw out the temporal analogue to Wittgenstein's remark in RFM about the expansion of π, we might indeed be tempted to say something like: even God can perceive and understand the facts about things happening in time only on the basis of an activity that is itself

[20] For of course Wittgenstein's point here is exactly not to invoke or rely upon conditions of an interior, psychological, or subjective kind as an alternative to the Platonism he also rejects. It is rather to criticize the whole configuration of thought about mathematics and time that is marked by the oscillation between Platonism and subjectivism, and in particular determined by the conception of the nature of a rule that both essentially share.

temporal, i.e. itself *within* the time it perceives and understands. To say this would not be to deny the very possibility of something like a view on the *whole* of time (a view, as it were, of the world *sub specie aeternitatis*). But it would be to deny that such a view can be taken *otherwise* than from a position that is itself essentially located within time, and to affirm that this necessity marks the very form of the givenness and availability of entities and phenomena themselves. To put the position this way would be to adumbrate one of the several ways in which a constitutive reflection on finitude and the infinite, including especially their forms in relation to "mathematical" truths and results, sheds light on the necessary form of the phenomena and experiences of the world itself, and indeed as they are in themselves.

What would be, then, the consequence of such a view, if maintained generally, for our understanding of the time of the world itself? In arguing against the possibility and tenability of the view of mathematical entities as simply *atemporal* and independent of the temporality and dynamism involved in a technique, Wittgenstein also suggests the emptiness of any idea of the world and its phenomena as surrounded by, or surveyable from, a position that is itself wholly extra-temporal. Such a position is the one that is envisioned as occupied, on the traditional "Platonist" conception, by the mathematical entities itself, and the idea of a divine intellect capable of knowing them in their infinite totality without (temporal) calculation or procedure then supplies the necessary, if falsifying, correlative position of possible knowledge. But if the picture of mathematical truth and entities that situates them in such a position is shown, on the basis of a constitutive consideration of the temporality of mathematical technique, to be untenable, then it is doubtful whether any significant support remains for the idea of an extra-temporal realm of determinate existence at all. Since this idea has, in fact, drawn much of its historical support (both in the philosophy of Plato himself and in subsequent developments of the theme) from the putative "timelessness" of mathematical entities, once this support is removed, it is not clear whether the overall conception of temporality that it itself involves can remain in place.[21] At any rate, since it is not clear any longer that *mathematics* requires, or even encourages us, to postulate or invoke any kind of "timeless" realm of existence outside the (temporal) world, it seems that any considerations that still incline us to do so must be seen to arise from very different (and distinctively non-mathematical) sources.

[21] For a fuller argument, drawing in part on Wittgenstein, and a development of the alternative temporality of the world this might be taken as suggesting, see Livingston (2017), esp. chapters 6 and 7.

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